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Endogenous sunspots, pseudo-bubbles, and beliefs about beliefs

Alan Kraus[a,]*, Maxwell Smith[b]

[a] *Faculty of Commerce, University of British Columbia, 2053 Main Mall, Vancouver, BC Canada V6T 1Z2*

[b] *Prudential Investments, Canada*

Abstract

We analyze a simple sunspot model that represents a standard securities market without sidebets on an exogenous sunspot event. The multiple self–fulfilling equilibria that arise in the model are based on investors' uncertainty about other investors' beliefs. Hence, these multiple equilibria are 'endogenous sunspots'. We show that endogenous sunspots can arise even with complete markets to which all investors have access and homogeneous beliefs, provided the homogeneity is not common knowledge. We also show that endogenous sunspots can produce 'pseudo–bubbles' in which the risky asset price is higher (or lower) at all dates than in a no–sunspot equilibrium.

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1. Introduction

Large and rapid movements in securities prices, such as those of October 19–20, 1987, are dramatic events for which several possible explanations have been offered. One explanation is that these market movements, in fact all market price movements, reflect the arrival of new information about earnings prospects and other fundamental factors affecting the dividend stream. There is little question that such information plays a major role in much market movement

* Corresponding author. Tel: (604) 822-8342; fax: (604) 822-8521; e-mail: alan.kraus@commerce.ubc.ca.

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but many market observers question the notion that this explanation covers cases such as the October 1987 crash, for example. At the other extreme is the explanation that fads and fashions are an important factor underlying securities prices and that large price movements may result from major changes in fads.

The many times that stock prices have changed by large amounts without any accompanying news that might explain the price move has led many to believe that stock markets are inherently emotional. Keynes in his classic writings expressed this intuition with his notion of animal spirits or market psychology affecting asset prices. More recently, financial economists have investigated more rigorously the nature of stock market volatility. The paper by Shiller (1981) for example has led to a growing literature trying to determine whether stock prices are too volatile relative to the volatility of their fundamental value. The related idea that trading creates volatility was given empirical support in French and Roll (1986) who found that prices are more volatile when markets are open for trading.

Economists starting with Cass and Shell have investigated asset pricing models that contain a role for market psychology or more generally extrinsic uncertainty referred to as 'sunspots'. Although the seminal article on sunspot equilibria by Cass and Shell (1983) begins with reference to the stock market, sunspot equilibria have not played a significant role in models of securities markets in the financial economics literature. Rational expectations models incorporating sunspot equilibria have either not featured financial securities other than money or, in the case of the Cass–Shell model, involved securities whose payoffs are contingent on the occurrence of the exogenous event called 'sunspots'. In other words, the financial securities available in the Cass–Shell sunspot equilibrium are sidebets about the occurrence of sunspots. Such sidebets on extraneous events are not, of course, a significant factor in actual securities markets.

In this paper we consider a simple 'sunspot' model that represents standard financial assets where the role of market psychology or extrinsic uncertainty is played by beliefs about beliefs. We show that beliefs about beliefs are capable of explaining how trading by rational investors can create volatility that is unrelated to changes in fundamental value. The beliefs about beliefs equlibrium we consider in this paper fulfils the intuition of market psychology because the equilibrium in a sense pulls itself up by its own bootstraps. Investors in the model must believe that the equilibrium exists for it to exist. If even some investors believe that it might exist then it can exist.

One of the purposes of this paper is to explore the existence of multiple self-fulfilling equilibria in securities markets when the securities' payoffs are not contingent on some extraneous event unrelated to the fundamentals of the economy (i.e., preferences or endowments). That is, the securities in our model include the sorts of risky and riskless assets that figure in usual securities market models but exclude explicit sidebets on extraneous events, i.e., on sunspots.

Consequently, the multiple self-fulfilling equilibria that arise in our model are based on factors endogenous to the market; hence, we call them 'endogenous sunspots'. The factor that gives rise to endogenous sunspots in our model is uncertainty about the beliefs of other market participants.

In our model, markets are complete and it is common knowledge that individuals share common beliefs about payoff-relevant events. Individuals' beliefs about the endogenous-sunspot event are not common knowledge, however. We show that endogenous-sunspot equilibria can occur when there is uncertainty concerning aggregate beliefs about the sunspot event. In this situation, securities prices may be affected by beliefs about the endogenous-sunspot event when individuals actually share common beliefs about this event but the fact of common beliefs is not common knowledge.

Our pure exchange economy has three dates, of which the first two are times when securities are traded and the final date is when all security payoffs and consumption occur. All investors act as price takers in the model. There is a riskless security that serves as numeraire and a risky security. The item of particular interest in the model is the price of the risky security at the second trading date, especially how it is affected by investors' beliefs about it. The state of beliefs about this price acts as the event that generates the possibility of endogenous-sunspot equilibria. These equilibria exist because of investors' uncertainty about the beliefs of other investors about beliefs about the future price of the risky asset.

Because we wish to abstract from the effects of private information about payoff-relevant events, we assume that it is known that everyone has the same information about the payoff of the risky security. This immediately raises the question of why investors would care about the beliefs of other investors in choosing their portfolios. If we considered a one-period model in which returns depended only on final security payoffs, the answer is that they wouldn't.

Our model has two trading dates before security payoffs occur, so the return from holding the risky security at the first date depends on its price at the second date. Investors base their initial trading date portfolio choices on their beliefs about the price at the second trading date. The model contains investors with different risk preferences, so the conditions for aggregation in Brennan and Kraus (1978) and Rubinstein (1974) are not present. As a result, the price at the second trading date depends on the portfolio choices investors make at the initial trading date. However, the only factor that could lead to one price as opposed to another at the second trading date is differences in beliefs about that price, which means differences in beliefs about beliefs since nothing else is unknown.

Here is what we envisage. Each investor, being atomistic, has a belief about the probability distribution of the second trading date price. This can be thought of as a reduced form of that investor's beliefs about the portfolio choices other investors make based on their own beliefs. Different investors' beliefs may be similar or they may not; the only information an investor gets that reveals anything about others' beliefs is the initial trading date price. Investors choose initial trading date portfolios based on their beliefs about the second trading date price, which in turn is dependent on their initial trading date portfolios.

One set of beliefs and prices that satisfies this scenario is for every investor to believe that the only possible price at the second trading date is the initial trading date price. If everyone believes that the price will not change, then everyone trades to his or her final portfolio at the initial trading date and nothing changes at the final trading date. This is analogous to the 'no trade theorem' in Milgrom and Stokey (1982). This no-sunspot equilibrium is one possible equilibrium, but not the only one. More interesting equilibria also exist that involve situations in which the price does change at the second trading date and further trading occurs. These endogenous-sunspot equilibria exist because investors are uncertain about what other investors are thinking – in other words, because of beliefs about beliefs.

If investors are uncertain about the future price, it is because they are uncertain about the beliefs of other investors. Since investors are not homogeneous in risk preference, the second trading date price is affected by the distribution of securities across investors resulting from the portfolios chosen at the initial trading date. Since the latter depend on investors' beliefs about the second trading date price, an investor's uncertainty about the beliefs of other investors implies uncertainty about the future price. Our model presents equilibria in which this uncertainty is not resolved by observing the current price. In these endogenous-sunspot equilibria, investors' beliefs about beliefs are the sole cause of the price movement between the two trading dates.

The way the endogenous-sunspot equilibrium appears to individual investor i is as follows. Investor i believes that the beliefs of investors in the economy have one of two configurations and investor i attaches some probability to each configuration. One of the configurations of beliefs is consistent with a price P_a at the second trading date and the second configuration is associated with a price P_b. Therefore, these possible prices are used by investor i in choosing a portfolio at the initial trading date.

The implication of the rational beliefs character of the endogenous-sunspot equilibrium is that if the portfolio choices of all investors are made as described above, with each investor believing in the same two configurations of beliefs but not necessarily attaching the same probabilities to the likelihood of each configuration, and if beliefs are actually those of the first configuration, then when investors choose optimal portfolios at the final trading date the stock price will indeed be P_a. Similarly, if beliefs are actually those of the second configuration, then the final trading date stock price will indeed be P_b. In other words, beliefs are rational because they are self-fulfilling.

Two aspects of the endogenous-sunspot equilibria are particularly interesting. The first concerns how the possible second trading date prices relate to the price in the no-sunspot equilibrium. One possibility is for initial trading date beliefs to reflect a probability distribution of the future price with support on both sides of the no-sunspot price, i.e., in the endogenous-sunspot equilibrium investors believe that the future price may be higher or may be lower than the no-sunspot price.

Surprisingly, it is also possible for an endogenous-sunspot equilibrium to involve current and future prices which are higher with probability one in the beliefs of all investors than in the no-sunspot equilibrium, or for it to involve prices that are lower with probability one in the beliefs of all investors than in the no-sunspot equilibrium. In the former case, for example, the price at the initial trading date is higher than in the no-sunspot equilibrium and the price at the second trading date, although investors believe that it may be above or below the initial trading date price, is certain to be above the no-sunspot equilibrium price. Thus, the existence of an endogenous-sunspot equilibrium can lead to a raising (or a lowering) of security prices at all dates, relative to the absence of an endogenous-sunspot equilibrium. We call this a 'pseudo-bubble'. It should be noted that this effect does not represent a true bubble, however, since the price remains higher through the second trading date and ultimate security payoffs are not affected.

The second interesting aspect of endogenous-sunspot equilibria concerns a welfare comparison with the no-sunspot equilibrium. We show by numerical example that an endogenous-sunspot equilibrium may involve all investors having higher expected utilities, using their own beliefs, than they have under the no-sunspot equilibrium. In other words, self-supporting conjectures about future prices, in a market where investors can take positions to bet on their conjectures, may make everyone feel better off ex ante. A more typical result, however, is that replacing the no-sunspot equilibrium by an endogenous-sunspot equilibrium makes one group of investors better off and makes the other group worse off.

Our model can be characterized as a noisy rational expectations equilibrium. However, our model is sharply distinguished from other noisy rational expectations models in the fact that no investor in our model has information about security payoffs or endowments that other investors do not have. In other words, in our model you are interested in my belief but learning it would not change your belief about the final payoff. In previous models, it would. Previous noisy rational expectations models include those of Grossman–Stiglitz (1980), Hellwig (1980), Diamond–Verrecchia (1981), Admati (1985), Kyle (1989), Grundy–McNichols (1989), Allen et al. (1993) and Romer (1993).

Our endogenous-sunspot model is related, in name and substance, to the seminal idea of sunspot equilibrium introduced in Cass–Shell (1983). However, there are significant conceptual differences between our model and the economy analyzed in Cass–Shell. In their model, the existence of a sunspot equilibrium rests on one of three conditions. Either markets are incomplete, or not all individuals are allowed to trade in all markets (in the Cass–Shell overlapping generations model some individuals cannot trade in the securities market for sunspot sidebets because they are born after the sunsport event occurs), or individuals' beliefs about the sunspot event are heterogeneous and a sidebet is traded.

We show that an endogenous-sunspot equilibrium can occur even in the absence of the above conditions, (i.e. with complete markets, all individuals having access to all markets, and homogeneous beliefs.) The basic source of the difference is that the sunspot event in Cass–Shell is an exogenous event that will occur or not regardless of what happens in the market. In our model, on the other hand, the endogenous sunspot event is the revelation at the second trading date of the true state of investors' beliefs about other investors' beliefs. This difference has significant implications.

First, as noted earlier, the financial securities in the Cass–Shell model are cash and a sidebet on the occurrence of the exogenous-sunspot event. The sunspot sidebet is in zero net supply and its payoff does not depend on what happens in the economy. Rather, sidebet payoffs, assuming investors take nonzero positions in the sidebet, are what cause the equilibrium at the second trading date (the date when commodities are traded in the Cass–Shell model) to depend on the occurrence of the sunspot event. In our model, the financial securities are cash and an asset in positive net supply whose risky payoff at the final date determines the aggregate consumption in the economy. The market value of the risky asset at the second trading date, which plays a role in our model analogous to the payoff on the sunspot sidebet in the Cass–Shell model, clearly does depend on the nature of the equilibrium in the economy.

Second, the role of heterogeneous beliefs is different in our model compared to Cass–Shell. In the Cass–Shell model with complete markets, homogeneous beliefs prevent a sunspot equilibrium (i.e., an equilibrium in which prices are affected by the occurrence of sunspots) from occurring, regardless of whether the fact of homogeneous beliefs is or is not common knowledge. This is because, with homogenous beliefs, the market for the sunspot sidebet clears only when all investors hold zero positions in the sidebet, so the exogenous sunspot event has no effect on equilibrium at the second trading date.

In our model, as we demonstrate later, it is possible to have an endogenous-sunspot equilibrium with homogenous beliefs, provided the fact of homogeneous beliefs is not common knowledge. In other words, what is crucial in the endogenous sunspot model is not the actual degree of heterogeneity of beliefs but the degree of investors' uncertainty about the state of beliefs. An endogenous-sunspot equilibrium can exist when beliefs are homogeneous but investors do not know that this is the case and investors believe that other investors' beliefs may differ from their own. On the other hand, if homogeneous beliefs are common knowledge then these beliefs are degenerate (since they are beliefs about beliefs) and only the no-sunspot equilibrium exists.

The organization of the remainder of the paper is as follows. Section 2 contains the basic assumptions and notation. Section 3 deals with the equilibrium at the second trading date. Section 4 describes the initial trading date equilibrium. Section 5 presents numerical examples to illustrate the no-sunspot and endogenous-sunspot equilibria. Section 6 gives conclusions.

2. Assumptions and notation

There are three dates in our world ($t = 0, 1, 2$). In temporal order they are an initial trading date ($t = 0$), a final trading date ($t = 1$) and a security payoff date ($t = 2$). At the security payoff date one of two states, which we refer to as the 'payoff states', will be chosen by a random device ('nature'). Security payoffs depend on the payoff state that occurs. The probabilities of the two payoff states are common knowledge. For simplicity, we assume they are each 0.5.

Payoff state uncertainty is the only uncertainty that reflects fundamental randomness in our model, in the sense that there is no conceivable way of knowing before the security payoff date which of the two payoff states will occur. Payoff state uncertainty would be relevant even in a single-agent world. There are additional uncertainties from the viewpoint of an individual investor in the form of items about which the investor has imperfect information, although these items could be known if all investors were to pool their information. An example of the latter is uncertainty about other investors' beliefs. Clearly, this sort of uncertainty pertains only to multiple-agent economies. To emphasize the distinction, we refer to 'market states' to represent alternative possibilities in the investor's information where one of these possibilities already prevails but is not known by the investor. In other words, 'market states' represent what is not true randomness but rather imperfect knowledge by some or all investors. The possibility of alternative market states is what will give rise to endogenous-sunspot equilibria.

Investors in the model are atomistic price takers with preferences characterized by constant relative risk aversion. Those investors who have the same coefficient of relative risk aversion are grouped together and treated as an aggregate investor. Since investors may be heterogeneous in beliefs, an aggregate investor acts as though it were an investor with beliefs that depend on the beliefs of the various individual investors in that aggregate. Investors are assumed to hold rational beliefs, in a sense to be made precise below. No investor has private information about the beliefs of other investors, so no one has private information about the beliefs of aggregate investors. For simplicity, the specific model we analyze has all investors grouped into two aggregate investors ($n = 1, 2$).

The notation for security holdings and payoffs is summarized in Table 1. The market includes two securities: 'cash' pays one dollar in each payoff state and

Table 1
Notation. Holdings and payoffs

Initial trading round	Cash	Stock
Endowment		
Investor i	z_{ci}	z_{si}
Aggregate n	Z_{cn}	Z_{sn}
Chosen portfolio		
Investor i	y_{ci}	y_{si}
Aggregate n	Y_{cn}	Y_{sn}
Final trading round		
Endowment		
Investor i	y_{ci}	y_{si}
Aggregate n	Y_{cn}	Y_{sn}
Chosen portfilio		
Investor i	x_{ci}	x_{si}
Aggregate n	X_{cn}	X_{sn}
Payoffs		
Investor i	x_{ci}	$\begin{cases} x_{si}A & \text{(prob 0.5)} \\ x_{si}B & \text{(prob 0.5)} \end{cases}$
Aggregate n	X_{cn}	$\begin{cases} X_{sn}A & \text{(prob 0.5)} \\ X_{sn}B & \text{(prob 0.5)} \end{cases}$

a share of 'stock' pays A dollars in one payoff state and B dollars in the other payoff state. (Recall that the two payoff states are assumed to be equally probable). We let the subscripts c and s refer to cash and stock, respectively. There are two rounds of trading. We let z, y, and x denote an individual investor's initial endowment, her holding after the first trading round, and her holding after the second trading round, respectively. Similarly, Z, Y, and X denote the totals of z, y, and x, respectively, across all investors constituting an aggregate investor.

At each of the two trading dates an equilibrium price of a share of stock in units of cash is established to satisfy the market clearing conditions:

$$Z_c \equiv \Sigma_n Z_{cn} = \Sigma_n Y_{cn} = \Sigma_n X_{cn}, \tag{1a}$$

$$Z_s \equiv \Sigma_n Z_{sn} = \Sigma_n Y_{sn} = \Sigma_n X_{sn}. \tag{1b}$$

In order to derive the equilibrium for each trading date, we first develop the optimal portfolio choice conditions for an individual investor. As is usual, we work backwards in time, considering first the final trading date and then the initial trading date.

3. Final trading date equilibrium

All investors in our model have preferences over consumption characterised by constant relative risk aversion. We consider individual investor i, whose coefficient of relative risk aversion is g_i and who enters the final trading round with y_{ci} of cash and y_{si} of stock. Recall that the payoff at $t = 2$ to a share of stock is A or B, with equal probability. For convenience, we assume that $A > B$. At $t = 1$, the investor chooses final holdings x_{ci} of cash and x_{si} of stock to solve

$$\text{Max} \quad \frac{1}{2}\left(\frac{1}{1-g_i}\right)(x_{ci} + Ax_{si})^{1-g_i} + \frac{1}{2}\left(\frac{1}{1-g_i}\right)(x_{ci} + Bx_{si})^{1-g_i} \tag{2a}$$

$$\text{subject to} \quad x_{ci} + P_1 x_{si} = y_{ci} + P_1 y_{si} \equiv w_{1i}, \tag{2b}$$

where P_1 is the price of a share of stock, in units of cash, at the final trading date. Absence of arbitrage requires that $A > P_1 > B$.

The solution to Eqs. (2a) and (2b) is

$$x_{ci} = \left[\frac{A(A-P_1)^{-1/g_i} - B(P_1 - B)^{-1/g_i}}{(A-P_1)^{1-(1/g_i)} + (P_1 - B)^{1-(1/g_i)}}\right] w_{1i}, \tag{3a}$$

$$x_{si} = \left[\frac{-(A-P_1)^{-1/g_i} + (P_1 - B)^{-1/g_i}}{(A-P_1)^{1-(1/g_i)} + (P_1 - B)^{1-(1/g_i)}}\right] w_{1i}. \tag{3b}$$

Substituting the optimal holdings from Eqs. (3a) and (3b) into Eq. (2a), the derived utility of final trading date wealth for investor i is

$$V_i(w_{1i}; P_1) = \left(\frac{1}{1-g_i}\right)(A-B)^{1-g_i}[(A-P_1)^{1-(1/g_i)} + (P_1 - B)^{1-(1/g_i)}]^{g_i} w_{1i}^{1-g_i}. \tag{4}$$

By Walras' law, the equilibrium price is found by aggregating Eq. (3a) or Eq. (3b) over all investors and setting the result equal to the fixed supply to satisfy the market clearing condition. As noted earlier, we group together all individual investors who have the same relative risk aversion coefficient (same g value) and consider the group as an aggregate investor. For simplicity, we assume that there are only two aggregate investors, so $n = 1, 2$ in Eqs. (1a) and (1b). Aggregating Eq. (3b) over investors and applying Eqs. (1a) and (1b), the condition for equilibrium at $t = 1$ is

$$\begin{aligned} Z_s = &\left[\frac{-(A-P_1)^{-1/g_1} + (P_1 - B)^{-1/g_1}}{(A-P_1)^{1-(1/g_1)} + (P_1 - B)^{1-(1/g_1)}}\right](Y_{c1} + P_1 Y_{s1}) \\ &+ \left[\frac{-(A-P_1)^{-1/g_2} + (P_1 - B)^{-1/g_2}}{(A-P_1)^{1-(1/g_2)} + (P_1 - B)^{1-(1/g_2)}}\right](Y_{c2} + P_1 Y_{s2}). \end{aligned} \tag{5}$$

Clearly, P_1 depends on the holdings of the aggregate investors at the end of the initial trading round. Since a budget constraint must be satisfied at the initial trading round and initial endowments Z_{c1}, Z_{c2}, Z_{s1}, Z_{s2} are known, using the initial trading date share price, P_0, allows the other three Y variables to be determined when one of them, say Y_{s1}, is known. Thus,

$$Y_{c1} = Z_{c1} + P_0(Z_{s1} - Y_{s1}), \tag{6a}$$

$$Y_{c2} = Z_{c2} + P_0(Y_{s1} - Z_{s1}), \tag{6b}$$

$$Y_{s2} = Z_s - Y_{s1}. \tag{6c}$$

One sense in which the final trading date equilibrium in Eq. (5) is not reflective of actual securities markets is that actual securities markets do not have a final trading date. The critical aspect of the final trading date is that portfolio choice does not depend on beliefs about future prices; no capital gains or losses occur after the final trading round. In real markets there is always a future trading round, so portfolio choice always reflects beliefs about future prices. For our purposes, however, it greatly simplifies the model to have a final trading date as a fixed starting point in the backwards-in-time analysis of equilibrium.

The key requirement in a multiperiod version of the model would be that at least some portion of each period's new information reflects something entirely new, so that the probability distribution of prices never becomes stationary. In a stationary world, rational investors know that other rational investors (at least, those equally informed) hold the same beliefs, which leaves no role for beliefs about beliefs to affect prices. Our position is that the real world is not stationary, and that an important implication of non-stationarity is that investors cannot be certain about the beliefs of other investors about future prices. This implies that the current price depends on beliefs about future prices and so depends on beliefs about beliefs.

Adopting the simplification of a final trading date makes this behavior much more transparent and makes it much easier to analyze formally. Specifically, from Eq. (5) we have the dependence of the final trading date price on the portfolio choices made at the initial trading date. In the next section, we analyze how those portfolio choices depend on beliefs about the final trading date price and how this affects the initial trading date price.

4. Initial trading date equilibrium

The individual investor chooses a portfolio at $t = 0$ to maximize the derived utility of final trading date wealth in (4) based on beliefs about the future share price, P_1. What sorts of uncertainty might those beliefs reflect? It is clear that an individual's uncertainty about security payoff information to be revealed next period, or about the endowments or risk preferences of other investors, or about the total supplies of securities, can cause the individual to be uncertain about next period's price. For example, Mirman and Reisman (1988) analyze an economy where endowments and/or utilities may depend on the state of nature. In such situations, if different investors hold different beliefs about the unknown item(s), then investors' portfolio choices may also be influenced by beliefs about others' beliefs. However, such cases of the relevance of 'beliefs about beliefs' to the current period's equilibrium depend on underlying uncertainty about fundamentals in the economy (i.e., payoff information, endowments, preferences, etc.). As such, these cases are not sources of the endogenous-sunspot phenomenon we are interested in.

Even if a given investor knows the final payoff information, endowments, and preferences of all other investors, her beliefs about the portfolio choices of the other investors depend on her beliefs about their beliefs. The portfolio choices of the other investors are relevant to her choice because they determine the holdings with which the other investors enter the final trading round and so affect the final trading date price. In other words, the investor's beliefs about the beliefs of other investors are equivalent to beliefs about next period's stock price. Similarly, the beliefs of the other investors can be regarded as their beliefs about next period's stock price. In summary, what a given individual investor believes about the future stock price depends on what that investor thinks other investors believe about the future price.

As described above, the situation at the initial trading date leads naturally to the possibility of multiple rational expectations equilibria. One equilibrium – one that always exists – involves every investor believing that the final trading date price will equal the initial trading date price. Under these beliefs, an investor's derived utility of wealth in Eq. (4) is a constant independent of the portfolio chosen at the initial trading date and the optimal final trading date portfolio in Eqs. (3a) and (3b) is the same as if no trading occurred at the initial trading date. Alternatively, this equilibrium is the same as if all investors choose their final trading date portfolios at the initial trading date and no price change or trading takes place at the final trading date. In other words, a no-sunspot equilibrium exists when all investors believe that nothing will happen at the final trading date; and under those beliefs, nothing does.

The initial trading date equilibrium becomes considerably more complex as soon as we allow it to encompass investors' uncertainty about the beliefs of other investors, where those beliefs are about the beliefs of other investors, and so on. The crucial point is that the beliefs of a given investor, even when these beliefs concern the completely endogenous question of the beliefs of other investors about beliefs, affect the first investor's portfolio choices in the initial trading round and, thus, are relevant to other investors who act on beliefs about the stock price at the second trading date. An implication of this is that rational beliefs about beliefs can be mapped into beliefs about the future stock price, provided that each price outcome in the support of these beliefs corresponds to a final trading date equilibrium for some configuration of other investors' beliefs that satisfy the same property. In other words, beliefs must jointly satisfy a rational expectations condition.[1]

In formalizing these notions, we will confine our attention to a highly simplified belief structure. Specifically, we assume that every investor believes that only two outcomes are possible for the equilibrium at the final trading date and that these are the same two outcomes for every investor. This is equivalent to assuming that every investor believes that the final trading date stock price will be one of two values, P_a or P_b, but different investors may attach different probabilities to these two values. In addition to simplicity, the case of two outcomes has the advantage of similarity to models of exogenous sunspots, as in Cass–Shell (1983), in which the exogenous sunspot either occurs or does not occur. Since the equilibrium at the final trading date depends on investors' initial trading date portfolios, which depend on investors' beliefs, our assumption amounts to having every investor attach positive probability only to the same two collections of investors' beliefs which represent two market states.

Under these assumptions, the initial trading date portfolio choice problem of investor i is to choose y_{ci} and y_{si} to solve

$$\text{Max} \quad q_i V_i(y_{ci} + P_a y_{si}; P_a) + (1 - q_i) V_i(y_{ci} + P_b y_{si}; P_b) \tag{7a}$$

$$\text{subject to} \quad y_{ci} + P_0 y_{si} = z_{ci} + P_0 z_{si}. \tag{7b}$$

where q_i denotes investor i's probability that $P_1 = P_a$. The optimal holding of stock in the solution to Eqs. (7a) and (7b) is

$$y_{si} = \left[\frac{-q_i^{-1/g_i}(P_a - P_0)^{-1/g_i} v_i(P_a)^{-1/g_i} + (1 - q_i)^{-1/g_i}(P_0 - P_b)^{-1/g_i} v_i(P_b)^{-1/g_i}}{q_i^{-1/g_i}(P_a - P_0)^{1-(1/g_i)} v_i(P_a)^{-1/g_i} + (1 - q_i)^{-1/g_i}(P_0 - P_b)^{1-(1/g_i)} v_i(P_b)^{-1/g_i}}\right] (z_{ci} + P_0 z_{si}), \tag{8a}$$

where

$$v_i(P) \equiv A^{1-g_i}[(A - P)^{1-(1/g_i)} + (P - B)^{1-(1/g_i)}]^{g_i}. \tag{8b}$$

In Eq. (8a), we have assumed, without loss of generality, that $P_a > P_b$, which implies (for absence of arbitrage) that $P_a > P_0 > P_b$.

[1] Mapping beliefs about beliefs into beliefs about the future price is equivalent to mapping beliefs about beliefs to beliefs about 'type'. This mapping allows us to collapse the infinite regress of 'beliefs about beliefs about beliefs about ...'. See Epstein and Wang (1996) for a general treatment of this point.

The belief q_i of investor i in Eqs. (7a) and (8a) regarding the probability that P_1 will have the value P_a is based on investor i's conjecture about the beliefs held by other investors. Every investor makes a corresponding individual conjecture about others' beliefs. Making these conjectures is not an extraneous exercise for investors: each investor must reflect some belief about the future price P_1 in her portfolio choice and each investor realizes that the future price depends on the portfolio choices of other investors which are based on their beliefs. Moreover, since investors face a unique situation every period, there is no reason for the conjectures of different investors about others' beliefs necessarily to be identical. What is required for a rational beliefs equilibrium, however, is that the beliefs of all investors be individually rational. That is, it must be the case that if other investors hold the beliefs that investor i believes they hold with probability q_i (respectively, $1 - q_i$), then the future price will indeed be P_a (respectively, P_b). And the corresponding condition must hold for the beliefs of every other investor. We turn now to examining how these requirements translate into the conditions for an endogenous-sunspot equilibrium.

It is convenient to consider the group of investors who make up one of the two aggregate investors, i.e., all those who have the same coefficient of relative risk aversion, say $g_i = g_n$. Taking this group as a single aggregate investor, called aggregate investor n ($n = 1, 2$), we let I_n denote the set of values of i for those individual investors who constitute aggregate investor n (i.e., those for whom $g_i = g_n$). The holding of stock of aggregate investor n after the initial trading round, denoted Y_{sn}, is the sum of y_{si} in (8a) over $i \varepsilon I_n$. Similarly, the initial endowments of cash and stock of aggregate investor n are given by $Z_{cn} \equiv \Sigma_{i \varepsilon I_n} z_{ci}$ and $Z_{sn} \equiv \Sigma_{i \varepsilon I_n} z_{si}$, respectively.

Since individual investors do not (necessarily) share common beliefs, an aggregate investor must be thought of as having 'beliefs' that are a composite of the beliefs of the individual investors. We say that Q is the composite belief in P_a of aggregate investor n consistent with demand Y_{sn} if Eq. (8a) holds with Y_{sn}, Z_{cn}, Z_{sn}, in place of y_{si}, z_{ci}, z_{si}, respectively, and Q in place of q_i:

$$Y_{sn} = \left[\frac{-Q^{-1/g_n}(P_a - P_0)^{-1/g_n} v_n(P_a)^{-1/g_n} + (1-Q)^{-1/g_n}(P_0 - P_b)^{-1/g_n} v_n(P_b)^{-1/g_n}}{Q^{-1/g_n}(P_a - P_0)^{1-(1/g_n)} v_n(P_a)^{-1/g_n} + (1-Q)^{-1/g_n}(P_0 - P_b)^{1-(1/g_n)} v_n(P_b)^{-1/g_n}}\right] (Z_{cn} + P_0 Z_{sn}). \quad (9)$$

In order for Eq. (9) reasonably to define aggregate investor n's composite belief Q, it needs to be confirmed that a value of Q between zero and one satisfies Eq. (9). In fact, we can establish the slightly stronger result that Eq. (8a) is satisfied for all $i \in I_n$ if and only if Eq. (9) is satisfied for Q between zero and one. One direction is immediate: if Eq. (9) is satisfied with $0 < Q < 1$, then a trivial solution to Eq. (8a) for all $i \in I_n$ is

$$q_i = Q, \tag{10a}$$

$$y_{is} = \left(\frac{Y_{sn}}{Z_{cn} + P_0 Z_{sn}}\right)(z_{ci} + P_0 z_{si}). \tag{10b}$$

In other words, given an aggregate investor with composite beliefs, one possibility supporting that aggregate is investors with homogeneous beliefs identical to the composite beliefs.

We sketch a proof of the other direction by induction. Suppose that Eq. (9) is satisfied with $0 < Q < 1$ for some subset of $i \in I_n$, so that Y_{sn}, Z_{cn}, Z_{sn} refer to aggregate values only over that subset, and that Eq. (8a) is satisfied for some $i \varepsilon I_n$ *not* in the subset. For convenience, define the function $f_n(q)$ as

$$f_n(q) = \frac{-q^{-1/g_n}(P_a - P_0)^{-1/g_n} v_n(P_a)^{-1/g_n} + (1-q)^{-1/g_n}(P_0 - P_b)^{-1/g_n} v_n(P_b)^{-1/g_n}}{q^{-1/g_n}(P_a - P_0)^{1-(1/g_n)} v_n(P_a)^{-1/g_n} + (1-q)^{-1/g_n}(P_0 - P_b)^{1-(1/g_n)} v_n(P_b)^{-1/g_n}}. \tag{11}$$

Thus, Eqs. (9) and (8a) for the subset of I_n and additional member of I_n, respectively, described above can be written as

$$Y_{sn} = f_n(Q)(Z_{cn} + P_0 Z_{sn}), \tag{12}$$

$$y_{si} = f_n(q_i)(z_{ci} + P_0 z_{si}). \tag{13}$$

Let the holding of stock by the larger aggregate investor consisting of the subset plus the additional investor, i.e., $Y_{sn} + y_{si}$, be written as

$$Y_{sn} + y_{si} = f_n(\hat{Q})[(Z_{cn} + z_{ci}) + P_0(Z_{sn} + z_{si})] \tag{14}$$

for some number $\hat{Q}$ where, from Eqs. (12) and (13),

$$f_n(\hat{Q}) = \left[\frac{Z_{cn} + P_0 Z_{sn}}{(Z_{cn} + z_{ci}) + P_0(Z_{sn} + z_{si})}\right] f_n(Q) + \left[1 - \frac{Z_{cn} + P_0 Z_{sn}}{(Z_{cn} + z_{ci}) + P_0(Z_{sn} + z_{si})}\right] f_n(q_i) \tag{15}$$

Differentiation of Eq. (11) shows that $f'_n > 0$. So from the mean value theorem applied to Eq. (15) it follows that $\hat{Q}$ lies between Q and q_i, which implies that $0 < \hat{Q} < 1$ and establishes the desired result.

The convenient feature of the above result is that we can analyze the initial trading date equilibrium in terms of the two aggregate investors and their composite beliefs. It is important to stress, however, that any individual investor knows only her own beliefs, *not* the beliefs of either aggregate investor (including the aggregate of which she is a part). For the equilibrium at the initial trading date to be consistent with each investor believing that the same two prices are possible at the final trading date, the initial trading date price must be consistent with two configurations of composite beliefs, $\{Q_{1a}, Q_{2a}\}$ and $\{Q_{1b}, Q_{2b}\}$, where the former configuration implies a final trading date price of P_a and the latter configuration implies a price of P_b.

Putting together our results, sufficient conditions for an endogenous-sunspot equilibrium at the initial trading date are that there be a stock price P_0, aggregate demands Y_{s1a}, Y_{s2a}, Y_{s1b}, Y_{s2b}, and composite beliefs Q_{1a}, Q_{2a}, Q_{1b}, Q_{2b} such that the solutions Y_{s1a} and Y_{s2a} to Eq. (9) for Q_{1a} and Q_{2a}, respectively, satisfy the market clearing condition (1b) and satisfy the final trading date equilibrium condition (5) for $P_1 = P_a$ and the solutions Y_{s1b}, Y_{s2b} to Eq. (9) for Q_{1b} and Q_{2b}, respectively, satisfy Eq. (1b) and satisfy Eq. (5) for $P_1 = P_b$. In other words, there must be two possible sets of beliefs about P_a and P_b (in fact, about the two sets of beliefs) such that if one set of beliefs is held the final stock price will turn out to be P_a and if the other set is held it will be P_b. Of course, only one of these sets of beliefs reflects the true situation but no investor is certain which. These conditions can be expressed as

4.1. *Conditions for endogenous-sunspot equilibrium with two possible sets of investors' beliefs*:

There exist initial stock price P_0, final stock prices P_a, P_b, composite beliefs (probabilities) $Q_{1a}, Q_{2a}, Q_{1b}, Q_{2b}$, and aggregate initial stock holdings $Y_{s1a}, Y_{s2a}, Y_{s1b}, Y_{s2b}$ such that

(1) $Y_{s1k} + Y_{s2k} = Z_s \; (k = a, b)$, (16a)

(2) $\{Y_{sn} = Y_{snk}, Q_n = Q_{nk}\}$ satisfies (9) $(n = 1, 2;\ k = a, b)$, (16b)

(3) $\{P_1 = P_k, Y_{s1} = Y_{s1k}, Y_{s2} = Y_{s2k},$

$Y_{c1k} = Z_{c1} + P_0(Z_{s1} - Y_{s1k}), Y_{c2} = Z_{c2k} + P_0(Z_{s2} - Y_{s2k})\}$ satisfies Eq. (5) $(k = a, b)$. (16c)

The solution to Eqs. (16a), (16b) and (16c) gives equilibrium prices and the initial trading date holdings Y_{cnk} and Y_{snk} $(n = 1, 2;\ k = a, b)$, chosen by the two aggregate investors. Their final trading date holdings, X_{cnkl} and X_{snkl} $(n = 1, 2;\ k = a, b;\ l = a, b)$, where k refers to the state of composite beliefs and l refers to the final trading date price, are given by summing (3) over all $i \in I_n$:

$$X_{cnkl} = \left[\frac{A(A - P_l)^{-1/g_n} - B(P_l - B)^{-1g_n}}{(A - P_l)^{1-(1/g_n)} + (P_l - B)^{1-(1/g_n)}}\right](Y_{cnk} + P_l Y_{snk}), \tag{17a}$$

$$X_{snkl} = \left[\frac{-(A - P_l)^{-1/g_n} + (P_l - B)^{-1/g_n}}{(A - P_l)^{1-(1/g_n)} + (P_l - B)^{1-(1/g_n)}}\right](Y_{cnk} + P_l Y_{snk}). \tag{17b}$$

It is important to note that Eqs. (17a) and (17b) includes cases of final trading date holdings that in fact cannot occur, *viz*., all cases where $l \neq k$. If the state of composite beliefs is k, then the final trading date market clears at price P_k and the price does not turn out to be P_l. Investors do not know the true state of composite beliefs at the initial trading date, however. Therefore, they contemplate at the initial trading date their future actions for final trading date prices P_a and P_b even though only the price that corresponds to the true state of composite beliefs will actually occur. This represents one difference between the endogenous-sunspot and exogenous-sunspot situations. In the latter, the sunspot event is determined by (random) nature at the second trading date. The cases in which $l \neq k$ in Eqs. (17a) and (17b) are not equilibria; rather they are analogous to off-equilibrium-path actions in game-theoretic models.

In the no-sunspot equilibrium, in which $P_0 = P_a = P_b$, investors trade to their final holdings all at once at the initial trading date. Hence, we can solve for this equilibrium by substituting in Eq. (5) Z_{cn} for Y_{cn} and Z_{sn} for Y_{sn}, respectively, for $n = 1, 2$, and a single price, $\hat{P}_0$, for P_0 and P_1:

$$Z_s = \left[\frac{-(A-\hat{P}_0)^{-1/g_1} + (\hat{P}_0 - B)^{-1/g_1}}{(A-\hat{P}_0)^{1-(1/g_1)} + (\hat{P}_0 - B)^{1-(1/g_1)}}\right](Z_{c1} + \hat{P}_0 Z_{s1})$$
$$+ \left[\frac{-(A-\hat{P}_0)^{-1/g_2} + (\hat{P}_0 - B)^{-1/g_2}}{(A-\hat{P}_0)^{1-(1/g_2)} + (\hat{P}_0 - B)^{1-(1/g_2)}}\right](Z_{c2} + \hat{P}_0 Z_{s2}). \tag{18}$$

Solving Eq. (18) for $\hat{P}_0$ gives the stock price that prevails at both the initial trading date and the final trading date in the no-sunspot equilibrium. To calculate the investors' portfolio holdings for the no-sunspot equilibrium, we substitute in Eqs. (3a) and (3b) $\hat{P}_0$ for P_1, z_{si} for y_{si} (to reflect that investors trade from initial endowments directly to final holdings) and sum over all $i \in I_n$. This gives

$$\hat{X}_{cn} = \left[\frac{A(A-\hat{P}_0)^{-1/g_n} - B(\hat{P}_0 - B)^{-1/g_n}}{(A-\hat{P}_0)^{1-(1/g_n)} + (\hat{P}_0 - B)^{1-(1/g_n)}}\right](Z_{cn} + \hat{P}_0 Z_{sn}), \tag{19a}$$

$$\hat{X}_{sn} = \left[\frac{-(A-\hat{P}_0)^{-1/g_n} + (\hat{P}_0 - B)^{-1/g_n}}{(A-\hat{P}_0)^{1-(1/g_n)} + (\hat{P}_0 - B)^{1-(1/g_n)}}\right](Z_{cn} + \hat{P}_0 Z_{sn}), \tag{19b}$$

where $\hat{P}_0$ is the solution to Eq. (18) and there are no k, l subscripts on the cash and stock holdings because all investors' beliefs are that the price will remain unchanged at $\hat{P}_0$.

In the next section, we analyze some numerical examples of endogenous-sunspot equilibria. This permits us to present examples of the pseudo-bubble phenomenon as well as an example in which all investors have higher ex ante expected utility in the endogenous-sunspot equilibrium than in the no-sunspot equilibrium.

5. Numerical examples

The maintained parameter values for the examples are in Table 2. The final payoff on a share of stock is 100 or 0 with equal probability. The economy contains individual investors of two types of constant relative risk aversion. Investors having a coefficient of relative risk aversion equal to 0.5 make up aggregate investor 1. Aggregate investor 2 comprises the remaining individual investors, each of whom has a coefficient of relative risk aversion equal to 2. The aggregate initial endowment of the investors composing each aggregate investor consists of 1 unit of cash and 1 share of stock.

As a benchmark, we first calculate the no-sunspot equilibrium for the example. Solving Eq. (18) yields the stock price $\hat{P}_0 = 1.163$ for this equilibrium and using this in Eqs. (19a) and (19b) gives the investors' holdings. These results are given in Table 3, along with the initial trading date expected utilities of the two aggregate investors. Recall that an aggregate investor has the same coefficient of relative risk aversion as each of its members and holds the aggregate of the initial endowments of its members. Since we make no assumptions about how individual endowments and individual beliefs are distributed across investors, we cannot specify the expected utility levels of individual investors. However, holding initial endowments fixed, a change in the equilibrium that changes the expected utility of an aggregate investor can be thought of as changing the

Table 2
Numerical example parameter values

Consumption date stock payoffs	$A = 100$ Aggregate investor 1	$B = 0$ Aggregate investor 2
Coefficient of relative risk aversion	$g_1 = 0.5$	$g_2 = 2$
Initial endowments		
Cash	$Z_{c1} = 1$	$Z_{c2} = 1$
Stock	$Z_{s1} = 1$	$Z_{s2} = 1$

Table 3
No-sunspot equilibrium

Stock price	$\hat{P}_0 = 1.163$ Aggregate investor 1	Aggregate investor 2
Equilibrium holdings		
Cash	$\hat{X}_{c1} = 0.025$	$\hat{X}_{c2} = 1.975$
Stock	$\hat{X}_{sl} = 1.838$	$\hat{X}_{s2} = 0.162$
Expected utility at initial trading date	13.717	-0.281

expected utility of the 'typical' individual investor in that aggregate in the same direction.

The conditions defining an endogenous-sunspot equilibrium are in Eqs. (16a) and (16b). Considering combinations of the two values of k and the two values of n, these conditions represent eight equations to be satisfied by the values of eleven variables (P_0, P_a, P_b, Q_{1a}, Q_{2a}, Q_{1b}, Q_{2b}, Y_{s1a}, Y_{s2a}, Y_{s1b}, Y_{s2b}). We resolve the three degrees of freedom by specifying values for the initial trading date stock price, P_0, and one set of the aggregate investors' composite beliefs, Q_{1a} and Q_{2a}. It will be convenient to regard (Q_{1a}, Q_{2a}) as the true composite beliefs, although no investor knows whether (Q_{1a}, Q_{1b}) or (Q_{2a}, Q_{2b}) are the true composite beliefs. After setting P_0, Q_{1a} and Q_{2a}, we use (16a) and (16b) to substitute for Y_{snk} and Y_{cnk} ($n = 1, 2$; $k = a, b$) in Eq. (16c) and obtain a system of four nonlinear equations in P_a, P_{b}, Q_{1b} and Q_{2b}. If Q_{1b} and Q_{2b} in the solution lie between zero and one, the solution is an endogenous-sunspot equilibrium.

We present three examples. Recall that in the no-sunspot equilibrium (Table 3) the stock price at both the initial and final trading dates is $\hat{P}_0 = 1.163$. The first endogenous-sunspot example illustrates a pseudo-bubble in which the stock price is above $\hat{P}_0$ at the initial trading date and remains about $\hat{P}_0$ at the final trading date in both market states. The second example shows that pseudo-bubbles (unlike rational, true bubbles) can also be negative; in this example the stock price is below $\hat{P}_0$ at the initial trading date and in both market states at the final trading date. The second example also illustrates the important point that an endogenous-sunspot equilibrium can exist with homogeneous beliefs provided that investors are not certain that belief are homogeneous. In this example beliefs in market state a are homogeneous but each individual investor assigns positive probability to the chance that the market state is b, in which beliefs are heterogeneous.[2] Thus, if the true market state is a, beliefs are actually homogeneous but investors are uncertain about this. The third example is a situation in which the initial trading date stock price in the endogenous-sunspot equilibrium, P_0, equals the initial trading date stock price in the no-sunspot equilibrium, $\hat{P}_0$. The interesting aspect of the endogenous-sunspot equilibrium in this example is that the ex ante expected utility of both aggregate investors is higher in both market states than it is in the no-sunspot equilibrium.

The results for the first endogenous-sunspot example are in Table 4. In market state a the beliefs of investors are such that the composite belief of aggregate investor 1 is 0.5 that the true market state is a and 0.5 that it is b, while the composite belief of aggregate investor 2 is 0.4 that the true market state is a and 0.6 that it is b. Market state b represents a collection of beliefs such that the

[2] As noted earlier, this situation bears some similarity to that in Allen–Morris–Postlewaite (1993), but differs significantly in that the investors in their model are uncertain about whether other investors have private information about final payoffs.

Table 4
Endogenous-sunspot equilibrium. Example 1

Inital trading date stock price	$P_0 = 1.300$ Aggregate investor 1	Aggregate investor 2
Initial trading date beliefs about final trading date stock price		
Market state a: $P_a = 1.384$	$Q_{1a} = 0.500$	$Q_{2a} = 0.400$
Market state b: $P_b = 1.229$	$Q_{1b} = 0.471$	$Q_{2b} = 0.517$
Equilibrium holdings		
Initial trading date		
Market state a		
Cash	$Y_{c1a} = -2.367$	$Y_{c2a} = 4.367$
Stock	$Y_{s1a} = 3.590$	$Y_{s2a} = -1.590$
Market state b		
Cash	$Y_{c1b} = 2.195$	$Y_{s2b} = -0.195$
Stock	$Y_{s1b} = 0.081$	$Y_{s2b} = 1.919$
Final trading date		
Market state a		
Price P_a		
Cash	$X_{c1aa} = 0.037$	$X_{c2aa} = 1.963$
Stock	$X_{s1aa} = 1.854$	$X_{s2aa} = 0.146$
Price P_b		
Cash	$X_{c1ab} = 0.025$	$X_{c2ab} = 2.197$
Stock	$X_{s1ab} = 1.644$	$X_{s2ab} = 0.175$
Market state b		
Price P_a		
Cash	$X_{c1ba} = 0.032$	$X_{c2ba} = 2.232$
Stock	$X_{s1ba} = 1.643$	$X_{s2ba} = 0.166$
Price P_b		
Cash	$X_{c1bb} = 0.029$	$X_{c2bb} = 1.971$
Stock	$X_{s1bb} = 1.843$	$X_{s2bb} = 0.157$
Expected utility at Initial trading date		
Market state a	13.395	− 0.266
Market state b	13.394	− 0.266

composite belief of aggregate investor 1 is 0.471 that the true market state is a and 0.529 that it is b and the composite belief of aggregate investor 2 is 0.517 that the true market state is a and 0.483 that it is b. Given these beliefs, market state a is associated with a final trading date stock price of 1.384 and market state b is associated with a stock price of 1.229.

The prices P_a and P_b are self-confirming. That is, if the true market state is a the portfolios chosen at the initial trading date based on the beliefs in market state a about the chances of P_a and P_b will be such that at the final trading date the stock price will indeed be P_a. Similarly, the positions chosen based on the beliefs in market state b will lead to a final trading date stock price of P_b.

The endogenous-sunspot equilibrium of Example 1 illustrates a pseudo-bubble. In the no- sunspot equilibrium (Table 3) the stock price is 1.163 at both trading dates. In Example 1, by contrast, the stock price is 1.300 at the initial trading date and is either 1.348 or 1.229 at the final trading date, depending on whether beliefs are those of market state a or market state b. Thus, in this endogenous-sunspot equilibrium the stock price is higher at all trading dates than it is in the no-sunspot equilibrium. However, as noted earlier, this pseudo-bubble situation is not, in fact, a bubble since it has no probability of ever 'bursting'.

The results of Example 2, shown in Table 5, illustrate two of the potential characteristics of endogenous-sunspot equilibria described earlier. One of these is the phenomenon of a negative pseudo-bubble. Although a rational bubble cannot be negative, the same is not true of an endogenous-sunspot equilibrium pseudo-bubble. In Example 2 the stock price both at the initial trading date and at the final trading date in both market states is below the no-sunspot equilibrium stock price.

The second phenomenon exhibited in Example 2 is particularly interesting because it illustrates a significant difference between an endogenous-sunspot equilibrium and an equilibrium based on an exogenous-sunspot event as in Cass–Shell (1983). As noted earlier, a sunspot equilibrium with complete markets can occur in the Cass–Shell setting only under heterogeneous beliefs about the exogenous-sunspot event. In Example 2, on the other hand, market state a represents homogeneous composite beliefs of the two aggregate investors about the chance of the market state being state a, although investors do not know that beliefs are homogeneous since they do not know whether the true market state is a or b. Thus, if the true market state is state a, then beliefs are, in fact, homogeneous.

The endogenous-sunspot equilibrium can exist even when beliefs are homogenous precisely because investors are not certain that beliefs are homogeneous. In the Cass–Shell model, on the other hand, the critical point about beliefs (in complete markets) is whether or not beliefs are homogeneous. With homogeneous beliefs a sunspot equilibrium does not exist in the Cass–Shell model regardless of whether or not investors are aware that beliefs are homogeneous. This difference underscores the point that the endogenous-sunspot equilibrium is a fundamentally different phenomenon from the Cass–Shell sunspot equilibrium.

Finally, the results of Example 3, shown in Table 6, illustrate that it is possible for an endogenous-sunspot equilibrium to produce a higher level of ex ante expected utility for both aggregate investors than does the no-sunspot equilibrium. For convenience in comparison, Example 3 is chosen so that the stock

Table 5
Endogenous-sunspot equilibrium. Example 2

Inital trading date	$P_0 = 0.600$ Aggregate investor 1	Aggregate investor 2
Initial trading date beliefs about final trading date stock price		
Market state a: $P_a = 1.102$	$Q_{1a} = 0.750$	$Q_{2a} = 0.750$
Market state b: $P_b = 0.158$	$Q_{1b} = 0.893$	$Q_{2b} = 0.170$
Equilibrium holdings		
Initial trading date		
Market state a		
Cash	$Y_{c1a} = 1.071$	$Y_{c2a} = 0.929$
Stock	$Y_{s1a} = 0.882$	$Y_{s2a} = 1.118$
Market state b		
Cash	$Y_{c1b} = -0.246$	$Y_{s2b} = 2.246$
Stock	$Y_{s1b} = 3.077$	$Y_{s2b} = -1.077$
Final trading date		
Market state a		
Price P_a		
Cash	$X_{c1aa} = 0.223$	$X_{c2aa} = 1.977$
Stock	$X_{s1aa} = 1.832$	$X_{s2aa} = 0.168$
Price P_b		
Cash	$X_{c1ab} = 0.002$	$X_{c2ab} = 1.065$
Stock	$X_{s1ab} = 7.636$	$X_{s2ab} = 0.257$
Market state b		
Price P_a		
Cash	$X_{c1ba} = 0.035$	$X_{c2ba} = 0.968$
Stock	$X_{s1ba} = 2.822$	$X_{s2ba} = 0.082$
Price P_b		
Cash	$X_{c1bb} = 0.001$	$X_{c2bb} = 1.999$
Stock	$X_{s1bb} = 1.518$	$X_{s2bb} = 0.482$
Expected utility at initial trading date		
Market state a	17.186	− 0.332
Market state b	16.490	− 0.313

price at the initial trading date in the endogenous-sunspot equilibrium is the same as in the no-sunspot equilibrium. As can be seen, the endogenous-sunspot equilibrium in Example 3 involves widely separated values of P_a and P_b. In qualitative terms, an endogenous-sunspot equilibrium that exhibits higher ex ante expected utility for both aggregate investors than in the no-sunspot equilibrium requires choosing parameter values in a relatively confined range. It would appear that this behavior is not a typical result for endogenous-sunspot equilibria.

Table 6
Endogenous-sunspot equilibrium. Example 3

Inital trading date	$P_0 = 1.163$ Aggregate investor 1	Aggregate investor 2
Initial trading date beliefs about final trading date stock price		
Market state a: $P_a = 3.131$	$Q_{1a} = 0.700$	$Q_{2a} = 0.200$
Market state b: $P_b = 0.498$	$Q_{1b} = 0.687$	$Q_{2b} = 0.246$
Equilibrium holdings		
Initial trading date		
Market state a		
Cash	$Y_{c1a} = -0.226$	$Y_{c2a} = 2.226$
Stock	$Y_{s1a} = 2.054$	$Y_{s2a} = -0.054$
Market state b		
Cash	$Y_{c1b} = -0.099$	$Y_{s2b} = 2.099$
Stock	$Y_{s1b} = 1.945$	$Y_{s2b} = 0.055$
Final trading date		
Market state a		
Price P_a		
Cash	$X_{c1aa} = 0.201$	$X_{c2aa} = 1.799$
Stock	$X_{s1aa} = 1.918$	$X_{s2aa} = 0.082$
Price P_b		
Cash	$X_{c1ab} = 0.004$	$X_{c2ab} = 2.064$
Stock	$X_{s1ab} = 1.592$	$X_{s2ab} = 0.271$
Market state b		
Price P_a		
Cash	$X_{c1ba} = 0.194$	$X_{c2ba} = 1.988$
Stock	$X_{s1ba} = 1.851$	$X_{s2ba} = 0.091$
Price P_b		
Cash	$X_{c1bb} = 0.0047$	$X_{c2bb} = 1.996$
Stock	$X_{s1bb} = 1.738$	$X_{s2bb} = 0.262$
Expected utility at initial trading date		
Market state a	13.817	-0.273
Market state b	13.802	-0.275

6. Conclusions

If one considers all the things on which the risky security price at the final trading date depends, we have purposely assumed in our model that there is no asymmetric information about any of the obvious items: information about final payoffs, security endowments, or risk preferences. While clearly heroic, this assumption allows us to focus on investors' beliefs about other investors' beliefs without those beliefs being reducible to beliefs about information payoffs, endowments, or preferences. In fact, our purpose is precisely to show that the initial and final security prices may depend on beliefs about beliefs even when there is no asymmetric information about anything else that affects prices. This is the essence of the endogenous-sunspot equilibrium.

Clearly, real life markets and real life price movements are driven by many forces in addition to the endogenous-sunspot effects we have illustrated above. Nonetheless, we argue that it is important for understanding speculative markets to realize that a significant element of uncertainty about future prices can exist, with investors acting rationally, simply because I'm uncertain about what you believe about my beliefs about your uncertainty about…

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References

Admati, A., 1985. A noisy rational expectations equilibrium for multi-asset markets. Econometrica 53, 629–657.

Allen, F., Morris, S., Postlewaite, A., 1993. Finite bubbles with short sale constraints and asymmetric information. Journal of Economic Theory 61, 206–229.

Brennan, M., Kraus, A., 1978. Necessary conditions for aggregation in securities markets. Journal of Financial and Quantitative Analysis 13, 407–418.

D. Cass, K. Shell, Do sunspots matter? Journal of Political Economy 91 (1983) 193–227.

Diamond, D., Verrecchia, R., 1981. Informational aggregation in a noisy rational expectations economy. Journal of Financial Economics 9, 221–235.

Epstein, L., Wang, T., 1996. Beliefs about beliefs' without probabilities. Econometrica 64, 1343–1373.

French, K., Roll, R., 1986. Stock market variances: the arrival of information and the reaction of traders. Journal of Financial Economics 17, 5–26.

Grossman, S., Stiglitz, J., 1980. On the impossibility of informationally efficient markets. American Economic Review 70, 393–408.

Grundy, B., McNichols, M., 1989. Trade and the revelation of information through prices and direct disclosure. Review of Financial Studies 2, 495–526.

Hellwig, M., 1980. On the aggregation of information in competitive markets. Journal of Economic Theory 22, 477–498.

Kyle, A., 1989. Informed speculation with imperfect competition. Review of Economic Studies 56, 317–356.

Mirman, L., Reisman, H., 1988. Price fluctuations when only prices reveal information. Economics Letters 27, 305–310.

Milgrom, P., Stokey, N., 1982. Information, trade and common knowledge. Journal of Economic Theory 26, 17–27.

Romer, D., 1993. Rational asset-price movements without news. American Economic Review 83, 1112–1130.

Rubinstein, M., 1974. An aggregation theorem for securities markets. Journal of Financial Economics 1, 225–244.

R. Shiller, Do stock prices move too much to be justified by subsequent changes in dividends? American Economic Review 71 (1981) 421–436.